Algebra and Number Theory

Junior League

1. The train Moscow–Petushkí covers the initial distance — from Moscow to Drézna — in three times longer time than from Leónovo to Petushki. Also, the distance from Drezna to Petushki is covered twice as fast as from Moscow to Leonovo. How many times longer is the path from Moscow to Petushki than the path from Drezna to Leonovo?

(None)

2. For which natural numbers n do there exist integers x, y, z such that

$$x + y + z = 0, \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{n}\ ?$$

(None)

3. Find all the perfect numbers which prime factorization satisfies the following property: every prime it contains is present in an odd power. A natural number is called perfect if it is equal to the sum all of its positive divisors smaller than the number itself — for example, $28 = 1 + 2 + 4 + 7 + 14$.

(None)

4. For natural n, k prove the inequality:

$$1 + \frac{1}{2} + \ldots + \frac{1}{nk} < \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) - \frac{1}{2}\,.$$

(None)

Senior League

1. Solve the inequality

$$20 - 14(20 - 14(20 - \ldots - 14(20 - 14x)\ldots)) > x.$$

There are 2014 pairs of parentheses in total.

(None)

2. Let $R(n)$ denote the number of ways to write a natural number n as the sum $n = a + b$ of two primes (for example, $R(3) = 0$, $R(4) = 1$, $R(10) = 3$, since

$10 = 3 + 7 = 5 + 5 = 7 + 3$). Show that if $p < q$ are two consecutive prime numbers, then the sum

$$2R(n-2) + 3R(n-3) + 5R(n-5) + \cdots + p \cdot R(n-p)$$

is divisible by q.

(None)

3. Find all the functions $f(x)\colon \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > \frac{f(n)+f(f(n))}{2}$ for all natural n.

(None)

4. Let $x_1 < x_2 < \ldots < x_n$ be an arithmetic progression of natural numbers, with x_1 and x_2 relatively prime. Let us assume that the product of the terms in the progression has $m < n$ prime divisors. Prove that $x_1^{n-m} \leqslant (n-1)!$

(None)